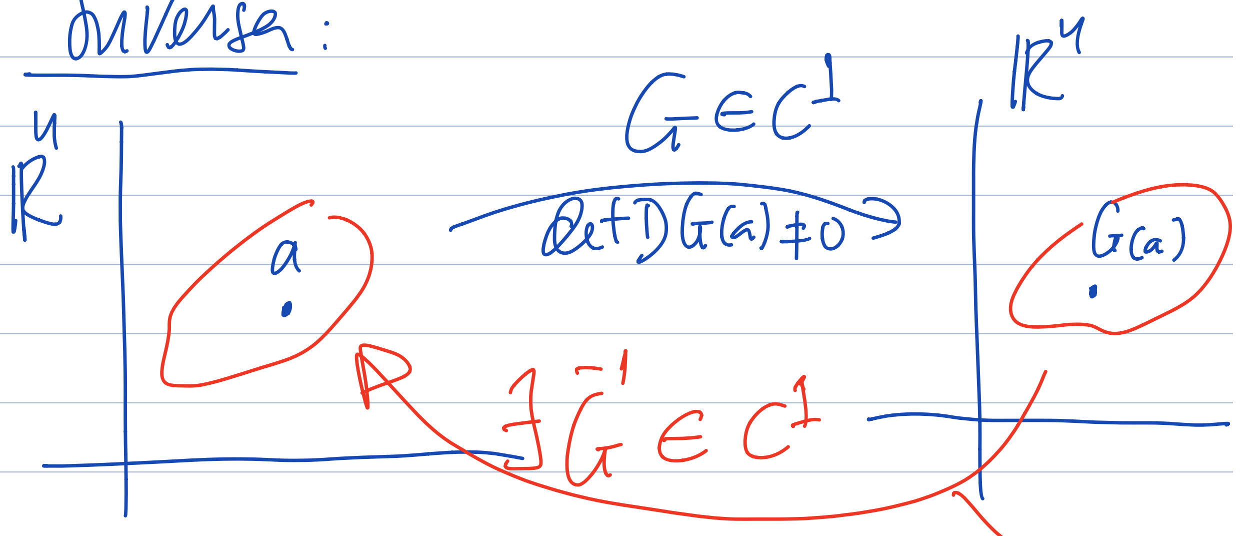
CDI-II - Prática 4/5/21

Ficha 8: Inversa, Implícita

Inversa:

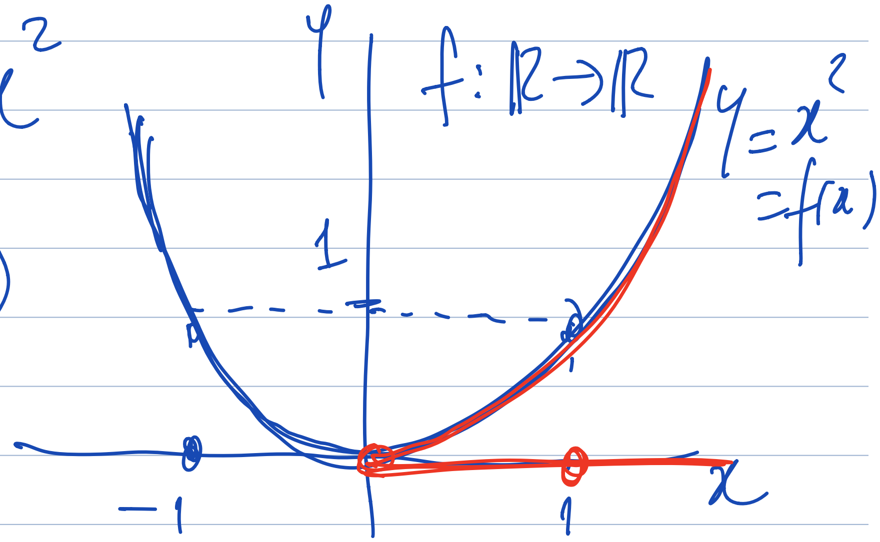


$$DG^{-1}(G(a)) = (DG(a))^{-1}$$

Inversa Local

$f(x) = x^2$

$f(-x) = f(x)$



$f(-1) = f(1) = 1$ $\qquad$ f não é injectiva em $\mathbb{R}$.

Se $x > 0$, $y = f(x) \Leftrightarrow x = f^{-1}(y)$
$= \sqrt{y}$

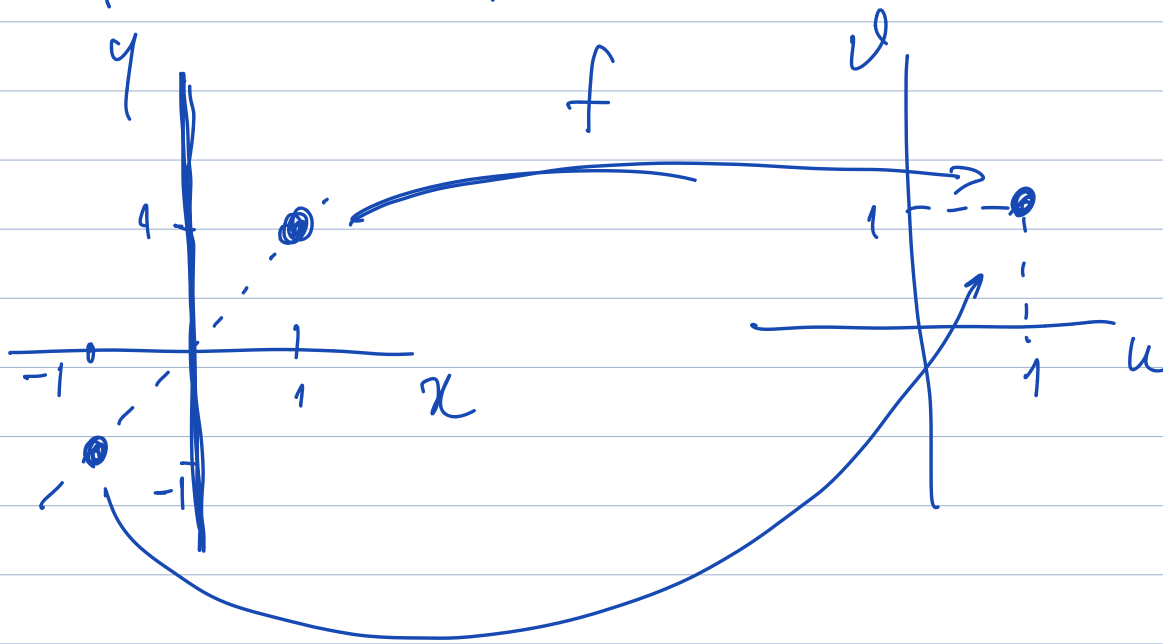
Se $x < 0$, $y = f(x) \Leftrightarrow x = f^{-1}(y)$
$= -\sqrt{y}$

1- a) $f(x,y) = \left(xy, \frac{y}{x}\right)$

$x \neq 0$ $= (u, v)$

$y = x \Rightarrow f(x,x) = (x^2, 1)$

$f(-1,-1) = f(1,1) = (1,1)$



f não é injectiva no seu domínio $\mathbb{R}^2 \setminus \{(0,y)\}$

1-b) $f(x,y) = (u,v)$

$\overset{?}{\Leftrightarrow} \quad (x,y) = f^{-1}(u,v)$

Contas!!!

$u \neq 0$

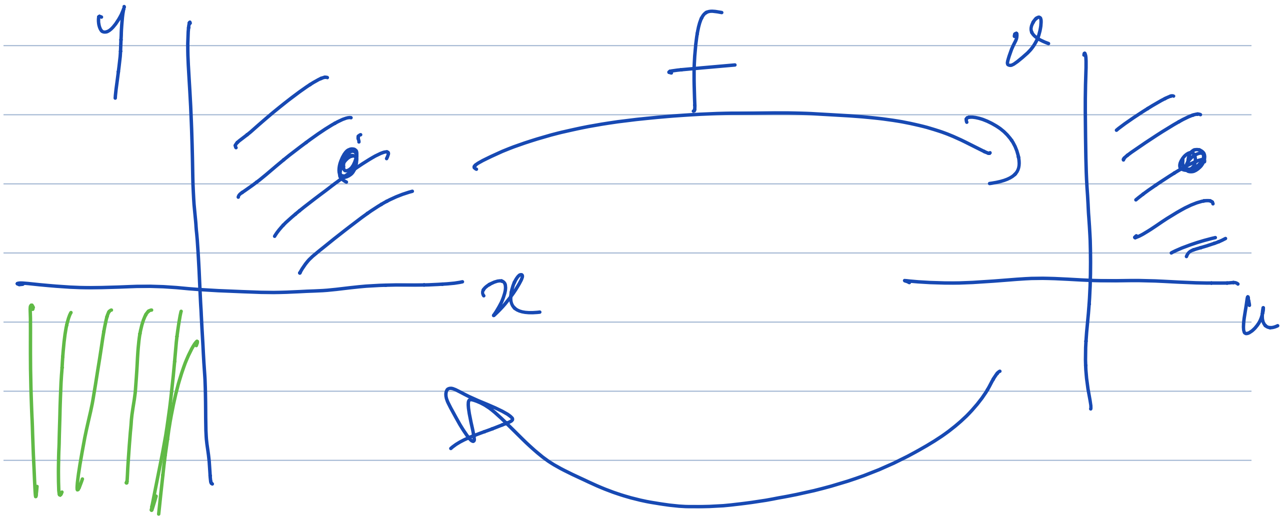
$$\begin{cases} u = xy \\ v = \dfrac{y}{x} \end{cases} \Leftrightarrow \begin{cases} u = x^2 v \\ y = xv \end{cases} \Leftrightarrow \begin{cases} x^2 = \dfrac{u}{v} \\ \text{---} \end{cases}$$

$x \neq 0$

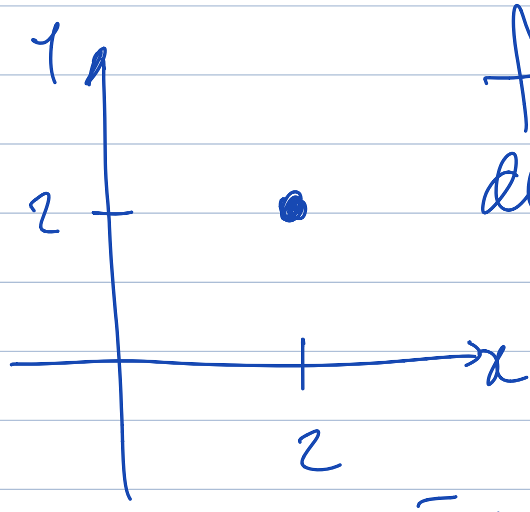
$$\Leftrightarrow \begin{cases} x = \sqrt{\dfrac{u}{v}} \\ y = \sqrt{\dfrac{u}{v}}\, v \end{cases} \vee \begin{cases} x = -\sqrt{\dfrac{u}{v}} \\ y = -\sqrt{uv} \end{cases}$$

$\sqrt{uv}$

u, v têm o mesmo sinal.



1-c) Inversa local (Teorema)



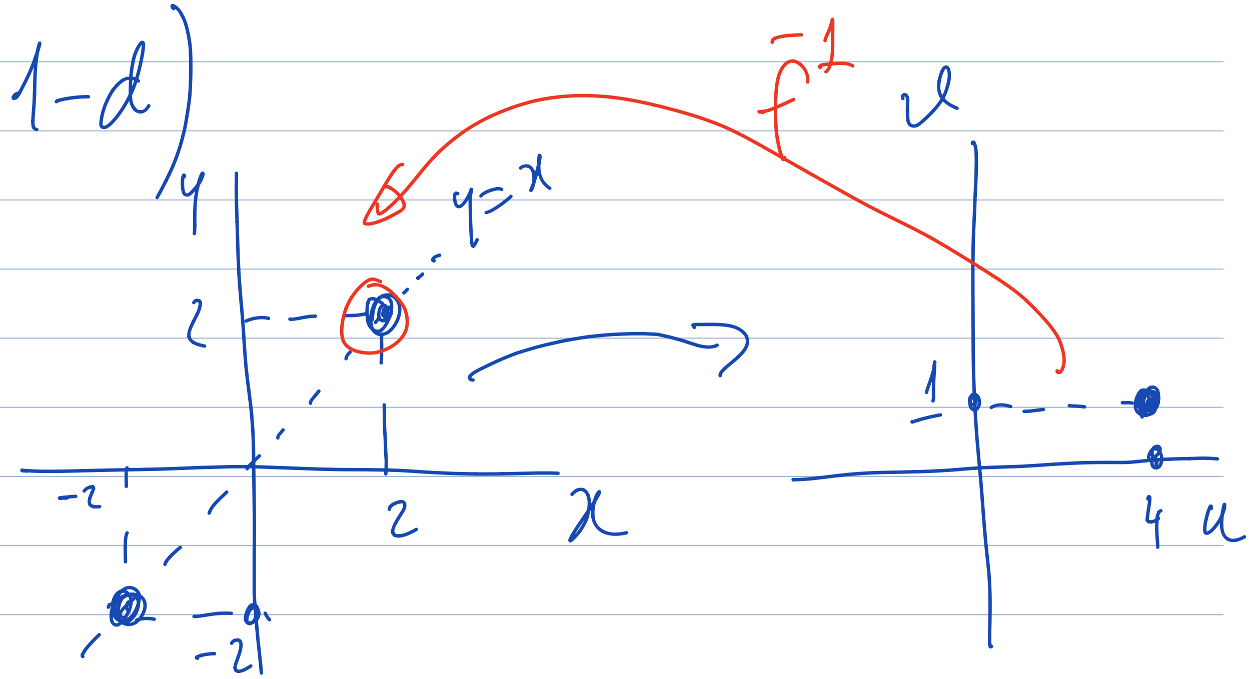
$f \in C^1$

$\det Df(2,2) \neq 0$

$$Df(x,y) = \begin{bmatrix} y & x \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{bmatrix}$$

$$\det Df(2,2) = \det \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = 2 \neq 0$$

1-d)



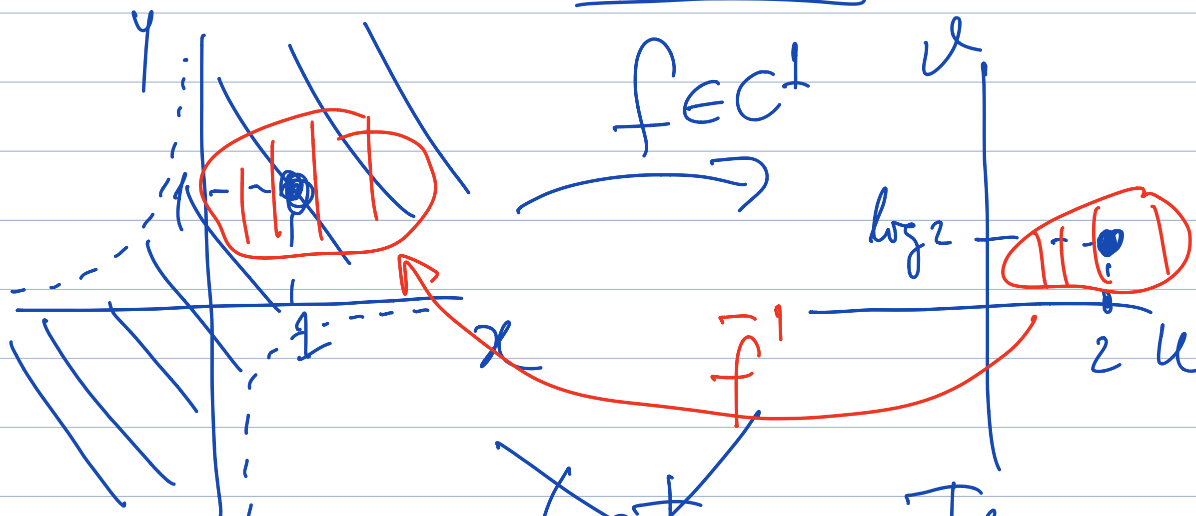
$$Df^{-1}(4,1) = \left(Df(2,2)\right)^{-1}$$

$$= \begin{bmatrix} 2 & 2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{2} & -2 \\ \frac{1}{2} & 2 \end{bmatrix} ✓$$

2 – $(x,y) \xrightarrow{f} (u,v)$

$$f(x,y) = \left(x+y+\operatorname{sen}(x-y),\ 1+\log(\underbrace{1+xy}_{>0})-x\right)$$

$$1+xy>0 \iff \boxed{xy>-1}$$



~~Contas~~ Teorema

$$(1,1) \xrightarrow{f} (2, \log 2)$$

$\det Df(1,1) \neq 0 \implies f^{-1}$ existe, C^1 "algures"

$$Df(x,y) = \begin{bmatrix} 1+\cos(x-y) & 1-\cos(x-y) \\ -1+\dfrac{y}{1+xy} & \dfrac{x}{1+xy} \end{bmatrix}_{2\times 2}$$

$$Df(1,1) = \begin{bmatrix} 2 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\det Df(1,1) = 1 \neq 0 \implies f^{-1} \text{ exists}$$

$$(x,y) = f^{-1}(u,v)$$

$$f(x,y) = (u,v)$$

$$Df(1,1) = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}_{(1,1)}$$

$$Df^{-1}(2, \log 2) = \left(Df(1,1) \right)^{-1}$$

$$(x, y) = f^{-1}(u, v)$$

$$\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \boxed{\dfrac{\partial y}{\partial v}} \end{bmatrix}_{(2, \log 2)}$$

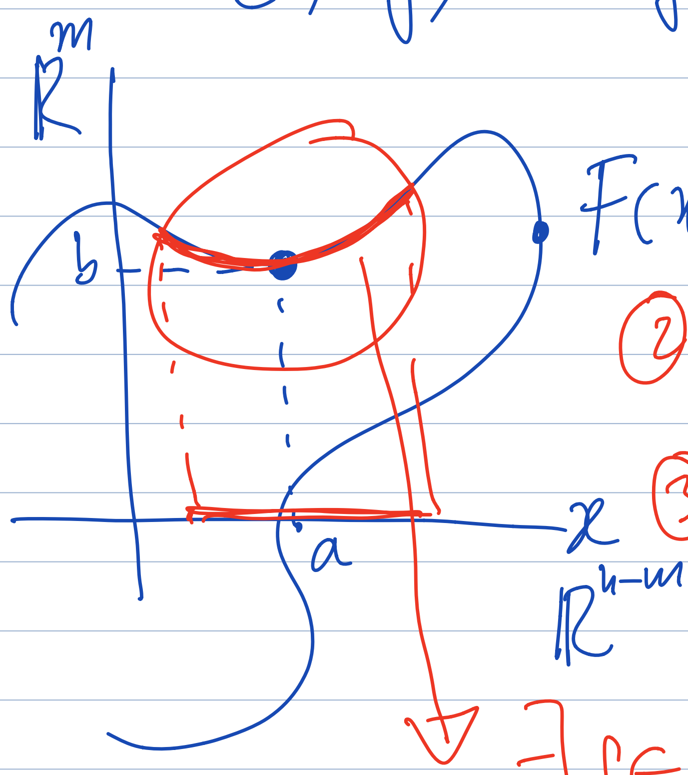
$$Df(1,1)^{-1} = \begin{bmatrix} 2 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & \boxed{2} \end{bmatrix}$$

$$\frac{\partial y}{\partial v}(2, \log 2) = 2 \quad //.$$

Implicita: (1) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $m<n$, $\mathcal{C}^1$

$\mathbb{R}^n = \mathbb{R}^{n-m} \times \mathbb{R}^m$
$(x, \; y)$

$x \to$ livres
$y \to$ dependentes.



$F(x,y) = 0$

(2) $F(a,b) = 0$

(3) $\det D_y F(a,b) \neq 0$

$\Rightarrow \exists f \in \mathcal{C}^1 : \boxed{y = f(x)}$

implicita

$$F(x, f(x)) = 0 \quad \boxed{\text{R. Cadeia}}$$

$$D_x F(a,b) + \underbrace{D_y F(a,b)}_{\det \neq 0} \boxed{Df(a)} = 0$$

$$Df(a) = -\left(D_y F(a,b)\right)^{-1} D_x F(a,b)$$

<u>Na prática</u>: Dados F, (a,b)

$$DF(a,b) = \left[\begin{array}{c|c} & \underbrace{\det \neq 0}_{m \times m} \end{array} \right]_{m \times n}$$

4 — $F(x,y,z) = 0 \overset{?}{\Longleftrightarrow} z = z(x,y)$ ou $z = f(x,y)$

$F: \mathbb{R}^3 \to \mathbb{R}$, C^1

$$DF(x,y,z) = \left[\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \quad \underbrace{\boxed{\frac{\partial F}{\partial z}}}_{\neq 0}\right](x,y,z)$$

$z = z(x,y)$

$F(x, y, z(x,y)) = 0$ R. Cadeia

$\frac{\partial z}{\partial x} = ?$

$$\frac{\partial F}{\partial x} + \underbrace{\frac{\partial F}{\partial z}}_{\neq 0} \boxed{\frac{\partial z}{\partial x}} = 0$$

$$\frac{\partial F}{\partial x}(x,y,z(x,y)) + \frac{\partial F}{\partial z}(x,y,z(x,y)) \frac{\partial z}{\partial x}(x,y) = 0$$

derivar em y

$$DF(0,0,1) = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix}$$

$z = z(x,y)$

—— || ——

5- $\begin{cases} F_1(x,y,z) = 0 & y^2+z^2-x^2-1=0 \\ F_2(x,y,z) = 0 & \text{----} \end{cases}$

$F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$, C^1

$x \longrightarrow (y(x), z(x)) \longrightarrow$

$y \longrightarrow (x(y), z(y))$

$z \longmapsto (x(z), y(z))$

$$DF(x,y,z) = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}_{2\times 3}$$

$$DF(0,1,0) = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

$\det = 2 \neq 0$

$y(x)$ e $z(x)$

$f(x) = (y(x), z(x))$

$f : \mathbb{R} \to \mathbb{R}^2$

$$\begin{cases} 0 + 2y' + 0z' = 0 \\ 1 + 2y' + z' = 0 \end{cases} \qquad \begin{cases} y'(0) = 0 \\ z'(0) = -1 \end{cases}$$